Elliptic Surfaces Constructed from Plane Quartic Curves

Daisuke Ibuki

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1. Introduction

A rational elliptic surface S is a smooth projective rational surface with a relatively minimal elliptic fibration $f : S \to \mathbf{P}^1$ over the base field k. We assume that the characteristic of k is either 0 or a sufficiently large prime number. With arithmetic applications in mind, we do not assume k to be algebraically closed, but in that case we fix its algebraic closure $\bar{k}$ and consider $\overline{S} = S \times_k \bar{k}$ for geometry. However, we always assume that f admits a section. Oguiso and Shioda [OS] classified all rational elliptic surfaces according to the structure of their Mordell-Weil lattices; there are 74 types in all.

There is a well-known relationship between plane quartic curves and rational elliptic surfaces. The double cover of $\mathbf{P}^2$ branched along a smooth quartic C is a del Pezzo surface V of degree 2. The pre-image of a pencil of lines centered at a point p is a pencil of elliptic curves on V. Blowing-up its base points suitably, we obtain a rational elliptic surface. We denote the elliptic surface constructed in this way by $\mathscr{E}_{C,p}$. (See §3 for more detail.)

Shioda [Sh2] used this construction to relate the 28 bitangents of a smooth plane quartic to the Mordell-Weil lattice of type E_7. Kuwata [K] classified all such rational elliptic surfaces $\mathscr{E}_{C,p}$ for a smooth plane quartic C.

Theorem 1.1 (Kuwata[K])**.** *If C is a smooth plane quartic curve, then a rational elliptic surface $\mathscr{E}_{C,p}$ falls into one of the six types, No. 1,2,3,4,6, or 13 in the table of Oguiso-Shioda* [OS].

In the sequel we call the table of Oguiso-Shioda "OS-Table" for short. We write OS#$(S) = n$ when the type of a rational elliptic surface S corresponds to No. n in the OS-Table.

In view of Kuwata's result we naturally ask a question: if we allow plane quartics C to be singular, and possibly reducible, then do all types of rational elliptic surfaces occur? Our answer is

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Theorem 1.2 (Main Theorem)**.** *For any number* $N \in \{1, 2, \dots, 74\}$*, there exisits a pair of a plane quartic curve* C *(possibly singular or reducible) and a point* p *in the plane such that* $\mathrm{OS\#}(\mathscr{E}_{C,p}) = N$*. Furthermore, if* $N = 62$*,* 64*, or* 73*, then* C *cannot be taken as an irreducible quartic.*

Examples of pairs (C, p) for each 74 type are listed in Table 14 and the resulting Weierstrass Forms in Table 15 in §9. The interested reader should download a Maple worksheet file, together with package for elliptic surfaces, from `http://c-faculty.chuo-u.ac.jp/~kuwata/Ibuki.html`.

Similar constructions have been done by Persson [P] in the case where the base field is the field of complex numbers. Also, the existence theorem has been proved by Shioda [Sh2]. However, neither Persson nor Shioda actually wrote down explicit examples, let alone Weierstrass equations. Our examples are all defined over a prime field at least of characteristic 0 or sufficiently large prime number.

This paper is organized as follows. First in §§2–3, we mention background materials about rational elliptic surfaces, plane curves, and the construction of $\mathscr{E}_{C,p}$. Then we consider the case where C is irreducible in §§4–5. In §4, we classify "the intersection sign" of a plane quartic C and a line L (Proposition 4.1), and in §5, we dispose all combinations of intersections $C \cap L$ and configurations of obtained fibers in $\mathscr{E}_{C,p}$ (Proposition 5.1). Then from observing the fibers, we deduce the latter conclusion of Theorem 1.2. In §6, we raise an example of a certain plane quartic C, and specifically illustrate our ideas for determining the OS-number of $\mathscr{E}_{C,p}$ by its root lattice when a position of $p \in \mathbf{P}^2$ is given. Next, we deal with the case where C is reducible in §7. In §8, we deal with the case where the OS-number of an elliptic surface is not uniquely determine by its root lattice, and explain a way to determine the OS-number of $\mathscr{E}_{C,p}$ through the use of the structure of the Mordell-Weil lattice. Then, finally in §9, we give a table including all 74 types of examples of rational elliptic surfaces constructed from specific plane quartics.

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2. 74 types of rational elliptic surfaces

Let S be a rational elliptic surface defined over k, with a relatively minimal elliptic fibration $f : S \to \mathbf{P}^1$. We always assume that f admits a section $O : \mathbf{P}^1 \to S$, and S has at least one singular fiber of f. Let E denote the generic fiber of f, which is an elliptic curve over the rational function field $K := k(t)$. Let $E(K)$ be the Mordell-Weil group of K-rational points of E, with origin O. Oguiso and Shioda completely classified the classes of rational elliptic surfaces into 74 types according to the structure of the triplet $(T, E(K)^0, E(K))$, where $E(K)$ is the Mordell-Weil lattice, $E(K)^0$ is the narrow Mordell-Weil lattice, and T is the direct sum of simple root lattices T_ν of type A, D, E. A simple root lattice T_ν corresponds to the singular fiber $F_\nu = f^{-1}(\nu)$ as in Table 1 (cf. [OS, §1]).

F_ν	$\mathrm{I_m}$	$\mathrm{I_m^*}$	II*	III*	IV*	IV	III	II
T_ν	A_{m-1}	D_{m+4}	E_8	E_7	E_6	A_2	A_1	A_0

TABLE 1.

Except five pairs (See §8), we can determine the OS-number of a rational elliptic surface by looking at the direct sum of simple root lattices.

3. Rational elliptic surface constructed from a plane quartic

In this section, we explain the construction of a rational elliptic surface from a plane quartic and a point on the plane.

Let C be a plane quartic curve (possibly singular or reducible), and let p be a point in $\mathbf{P}^2$, which may or not may be on the curve. We assume that p is not a singular point of C (Remark 3.1). Consider the pencil of lines Λ_p centered at p. We regard Λ_p as a line in the dual projective plane. Let $\pi_p : \mathbf{P}_0 \to \mathbf{P}^2$ be a blowing up at p. Naturally we obtain a $\mathbf{P}^1$-bundle $f_p : \mathbf{P}_0 \to \Lambda_p \cong \mathbf{P}^1$. On the other hand, let $\varphi_{F_C} : V_{F_C} \to \mathbf{P}^2$ be the double cover whose affine model is given by $w^2 = F_C(x, y, 1)$, where $F_C(x, y, z) = 0$ is a defining equation of C.

If we replace the equation F_C by αF_C with $\alpha \in k$, the resulting double covers $V_{\alpha F_C}$ and V_{F_C} are isomorphic over a quadratic extension of k. In other words, $\overline{V}_{\alpha F_C} \cong \overline{V}_{F_C}$ over $\bar{k}$. From now on until the end of this section we consider V_{F_C} over $\bar{k}$, and we write V instead of $\overline{V}_{F_C}$ for simplicity.

Now we use *Horikawa's canonical resolution* as follows (cf. [Ho]). We define p_i, ψ_i, C_i and E_i inductively as follows: We set C_0 to be the inverse images $\pi_p^{-1}(C)$. We define $\psi_i : \mathbf{P}_i \to \mathbf{P}_{i-1}$ to be the blowing up at p_{i-1}, where p_{i-1} is a singular point of C_{i-1}. Let E_i be the exceptional curve of ψ_i, and $m_{q_{i-1}}$ the multiplicity of C_i at q_{i-1}. We set $C_i = \psi_i^* C_{i-1} - 2[m_{p_{i-1}}/2]E_i$, where $[m_{p_{i-1}}/2]$ is the greatest integer not exceeding $m_{p_{i-1}}/2$. Then naturally we can obtain a double covering $\varphi_{C_i} : V_i \to \mathbf{P}_i$ ramified along C_i. We repeat this process until obtaining the nonsingular curve C_r.

We consider the following two cases.

(i) The case where p does not lie on C

In this case, C_0 does not contain the exceptional curve $E = \pi_p^{-1}(p)$. From Bezout's theorem, $C_0 \cap \tilde{L}$ consists of four points for a general fiber, where $\tilde{L} \subset \mathbf{P}_0$ is the strict transform of a general member $L \in \Lambda_p$. So the pull-back $\varphi_{C_0}^{-1}(\tilde{L})$ is a double cover of $\tilde{L}$ ramified at four points, which is a curve of genus one.

(ii) The case where p lies on C

In this case, C_0 contains the exceptional curve E, and we see that $C_0 = \tilde{C} + E$, where $\tilde{C}$ is the strict transform of C. Then, the strict transform $\tilde{L}$ intersects $\tilde{C}$ at three points and E at one point for a general fiber $\tilde{L} \subset \mathbf{P}_0$. Consequently, the pull-back $\varphi_{C_0}^{-1}(\tilde{L})$ is a curve of genus one as well as (i).

Thus, for both cases we obtain the elliptic fibration $f_p \circ \varphi_{C_0} : V_0 \to \Lambda_p \cong \mathbf{P}^1$. Then we blow up singular points and obtain the rational elliptic surface V_r and the elliptic fibration $\Phi_{C,p} := f_p \circ \psi_0 \circ \cdots \circ \psi_r \circ \varphi_{C_r} : V_r \to \Lambda_p \cong \mathbf{P}^1$.

The structure of rational elliptic surface obtained as above depends on a curve C and the position of a point p. Therefore we denote this rational elliptic surface by $\mathscr{E}_{C,p}$.

Remark 3.1. If we choose p a singular point of C, then $f_p \circ \varphi_{C_0} : V_0 \to \Lambda_P \cong \mathbf{P}^1$ is not an elliptic fibration. Indeed, a general fiber $\varphi_{C_0}^{-1}(\tilde{L})$ is not ramified at four points, hence it is not an elliptic curve.

Later we focus our attention on the singular fibers in $\mathscr{E}_{C,p}$ in order to construct all 74 types of rational elliptic surfaces in the OS-table. Then, when do singular fibers show up? For example, the pull-back of a tangent line of C by $\Phi_{C,p}$ is a singular fiber. Also, the pull-back of a line through a singular point of C is a singular fiber. Indeed, such a line meets C_0 at less than four points. In §5, we determine the configuration of the fiber by looking at the total transform $(\psi_0 \circ \cdots \circ \psi_r \circ \varphi_{C_r})^* L_0$, where L_0 is the inverse image of a line $L \in \Lambda_p$ by π_p.

We classify all patterns of the intersection points of a plane quartic and a line, and study the relation between the combination of intersection points and the fiber of $\mathscr{E}_{C,p}$. For a line L in Λ_p, we denote by F_L the fiber of $\mathscr{E}_{C,p}$ obtained from L.

4. Intersection of a given irreducible quartic and a line

In §§4–5, we consider the case where C is a plane irreducible curve. In this section, we classify the intersection points of a plane irreducible quartic curve and a line in $\mathbf{P}^2$.

First, we introduce three numerical invariants s_q, m_q, δ_q of a point q on a plane curve C. They are needed in classification of singularities of plane quartics.

- s_q Let C be a plane irreducible curve of degree n and $\phi : C_0 \to C$ be its normalization. For a point $q \in C$, we put

  $$\phi^{-1}(q) = \{q_1, \dots, q_s\} \text{ with } q_j \neq q_k \text{ for any } j \neq k,$$

  where s is the number of local irreducible branches at q. We put $s_q =: s$.

  If C is reducible, then we let $C = \sum C_k$ be it's irreducible decomposition, and we set $s_q := \sum s_{q_k}$, where q_k is a point on C_k.

- m_q For a curve C and a point $p \in C$, let f be a local equation of C at p. We define the *multiplicity* of C at q to be the largest integer r such that $f \in \mathscr{M}_q^r$, where $\mathscr{M}_q \subseteq \mathscr{O}_{C,q}$ is the maximal ideal. We let $m_q := r$. If C is a plane irreducible curve with the local equation f at q, then m_q is equal to the order of f at q.
- δ_q For a plane curve C, there exists a finite sequence of monoidal transformations with suitable centers $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \mathbf{P}^2$ such that the strict transform C_n of C on X_n is nonsingular (cf.[N, PART 2, 4.2]). Then, for a point p on C, we define δ_q by the formula

  $$\delta_q = \sum_{q'} \frac{1}{2} m_{q'}(m_{q'} - 1),$$

  where $\sum_{q'}$ runs over all infinitely near singular points q' lying over q (including q), and where $m_{q'}$ is the multiplicity of C_k at q.

It is known that each of s_q, m_q, δ_q is invariant under any change of local coordinate systems. We denote by $S_{(a,b,c)}$ a singular point with $s_q = a, m_q = b, \delta_q = c$.

Then we introduce the classification result of singularities of irreducible plane quartics. It is given as in Table 2 (cf. [N, PART 1, 2.2]).

sign	s_q	m_q	δ_q	name of q
N	1	1	0	nonsingular point
$S_{(1,2,1)}$	1	2	1	simple cusp of multiplicity 2
$S_{(1,2,2)}$	1	2	2	double cusp
$S_{(1,2,3)}$	1	2	3	ramphoid cusp
$S_{(1,3,3)}$	1	3	3	simple cusp of multiplicity 3
$S_{(2,2,1)}$	2	2	1	node
$S_{(2,2,2)}$	2	2	2	tacnode
$S_{(2,2,3)}$	2	2	3	osnode
$S_{(2,3,3)}$	2	3	3	tacnode-cusp
$S_{(3,3,3)}$	3	3	3	ordinary triple point

TABLE 2.

Our purpose is to check the singular fibers of a rational elliptic surface $\mathscr{E}_{C,p}$. For that purpose, it is reduced to observe the intersection points of a given quartic C and a line L in Λ_p because the pull-back of L by $\Phi_{C,p}$ is a fiber on $\mathscr{E}_{C,p}$. Therefore, it is important to investigate all configurations of C and L. More precisely for each sign in Table 2, we ask

- The possibilities of intersection multiplicity $(L \cap C)_q$, and
- whether a local branch of C at q is tangent to L or not.

We classify all patterns of a the intersection point of L and C.

Proposition 4.1. *For a line L meeting an irreducible plane quartic C at q, how C and L intersect at q is one of* 24 *patterns in* Table 3. ($(L \cap C)_q$ *expresses the intersection multiplicity at q.*)

We say these signs *"intersection signs at q"*.

Remark 4.1. We put * on the sign if L is tangent to C at q. If C has two or three local branches at q, we have several tangent lines for each branch. In these cases, we can distinguish by the number of the intersection multiplicity at q except for $S_{(2,3,3)}$. However, only $S_{(2,3,3)}$ has two different types of tangent lines but both of the intersection multiplicities are four. That is because $S_{(2,3,3)}$ has two local branch, one is nonsingular and the another is a simple

sign	N	N^{*1}	N^{*2}	N^{*3}	$S_{(1,2,1)}$	$S^*_{(1,2,1)}$
$(L \cap C)_q$	1	2	3	4	2	3
sign	$S_{(1,2,2)}$	$S^*_{(1,2,2)}$	$S_{(1,2,3)}$	$S^*_{(1,2,3)}$	$S_{(1,3,3)}$	$S^*_{(1,3,3)}$
$(L \cap C)_q$	2	4	2	4	3	4
sign	$S_{(2,2,1)}$	$S^{*1}_{(2,2,1)}$	$S^{*2}_{(2,2,1)}$	$S_{(2,2,2)}$	$S^*_{(2,2,2)}$	$S_{(2,2,3)}$
$(L \cap C)_q$	2	3	4	2	4	2
sign	$S^*_{(2,2,3)}$	$S_{(2,3,3)}$	$S^{*1}_{(2,3,3)}$	$S^{*2}_{(2,3,3)}$	$S_{(3,3,3)}$	$S^*_{(3,3,3)}$
$(L \cap C)_q$	4	3	4	4	3	4

TABLE 3.

cusp. If L is tangent to a nonsingular curve, the multiplicity is $2+2=4$, and if tangent to a simple cusp, the multiplicity is $3+1=4$. These are different forms (Figure 4).

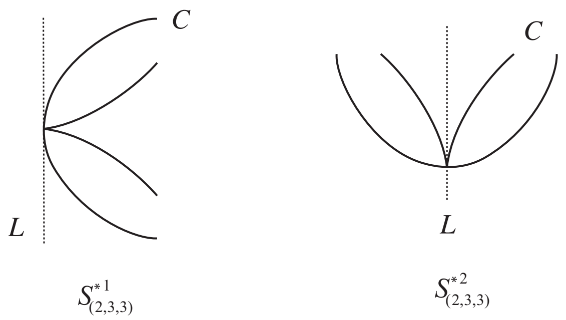

FIGURE 4.

5. Singular fibers in $\mathscr{E}_{C,p}$ for irreducible quartics

In this section, we consider the fiber F_L in $\mathscr{E}_{C,p}$. Let C be a plane irreducible quartic, p a point on the plane, and L a line through p. Let $q_1, \dots, q_n$ $(1 \le n \le 4)$ be the intersection points $C \cap L$. We denote by m_i the intersection multiplicity of C at q_i. It is obvious that the sum of m_i is four from Bezout's theorem. Then, we can display all intersection signs at each q_i from Proposition 4.1. For example, we denote by $S_{(2,2,1)}$-$S_{(2,2,1)}$ the combination where L meets two nodes of C. We say this combination *"the intersection combination"*.

Then, we can determine the configuration of the fiber F_L in $\mathscr{E}_{C,p}$ for the intersection combination according to the process of blowing ups. We give some examples as follows.

Notation 1. In the figures below, we denote a line not contained in each C_i by a dotted one.

Example 5.1. The intersection combination: $S_{(2,2,r)}$-N-$N(1 \leq r \leq 3)$.
In this case, we need to blow up r times $C_r \to C_{r-1} \to \cdots \to C_0$ (Figure 5). Taking the double cover ramified along C_r, we see the configuration of the fiber in $\mathscr{E}_{C,p}$ is $\mathrm{I_{2r}}$ by Kodaira's notation. Indeed, it turns that each of $\varphi_{C_r}^{-1}(\tilde{L})$ (where $\tilde{L}$ is the strict transform of L) and $\varphi_{C_r}^{-1}(E_r)$ is $\mathbf{P}^1$ by Rieman-Hurwitz formula, and $\varphi_{C_r}^{-1}(E_j)(j \neq r)$ consists of two components.

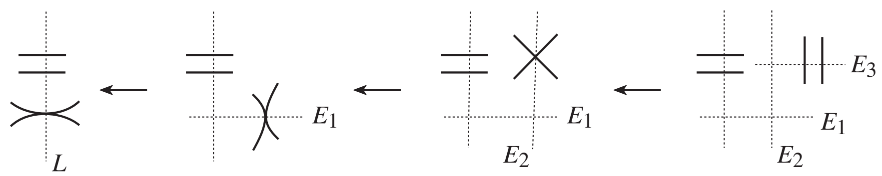

FIGURE 5. The case $r = 3$.

Example 5.2. The intersection combination: $S^*_{(2,2,3)}$
In this case, we blow up at $S_{(2,2,3)}$, and C_1 has again a singular point $p_1 = S_{(2,2,2)}$. The exceptional curve E_1 is not contained in C_1 because m_{p_1} is even. Of course p_1 is the intersection $\tilde{L} \cap E_1$. Hence, C_1 meets both of L and E_1 transversally at p_1. Indeed, if either $\tilde{L}$ or E_1 were tangent to C_1 at p_1, we would have the intersection multiplicity $(\tilde{L} + E_1).C_1 = L.C = 6$. Now we blow up at p_1 and get C_2 which has again a singular point $p_2 = S_{(2,2,1)}$. Lastly we have one more blow up at p_2 and get nonsingular curve C_3 (Figure 6).

By using a classification of singularities on irreducible plane quartics (cf. [N, PART 1, 2.2]), we obtain the following

Proposition 5.1. *The intersection combinations for an irreducible plane quartic C and a point p on $\mathbf{P}^2$ fall into* 38 *types in the following table. Moreover, for each intersection combination, the types of singular fibers in $\mathscr{E}_{C,p}$ are determined.*

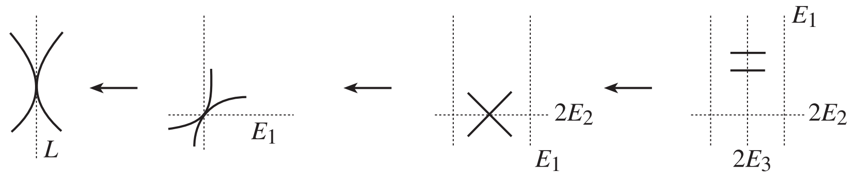

FIGURE 6. Taking the double cover ramified along C_3, we see the configuration of the fiber in $\mathscr{E}_{C,p}$ turns to be I_2^* in the same manner as Example 5.1.

Remark 5.1. For the combinations which contain N^*, we remark that whether p is on N^*. That is because we must blow up at p for construct $\mathscr{E}_{C,p}$ and the exceptional curve E_p is contained in the branch locus of the double cover.

Theorem 5.2. *Let C be a plane quartic and p be a point on $\mathbf{P}^2$. Any rational elliptic surface of* No.62,No.64,No.73 *in OS-table cannot be written as the form $\mathscr{E}_{C,p}$ provided C is irreducible.*

Proof. • No.62, No.64
The OS-table implies that root lattice T of a rational elliptic surface of No.62 (resp.64) is of type E_8 (resp. D_8), which corresponds to the singular fiber of type I_4^* (resp II^*). However we get a contradiction due to Table 7.

• No.73
The root lattice T of rational elliptic surface of No.73 is of type $D_4^{\oplus 2}$, which corresponds two singular fibers of type I_0^*. Then to obtain these singular fibers, we must have C which has two singular points of $S_{(3,3,3)}$ or $S^*_{(2,2,2)}$ due to Table 7. However we see that there exists no irreducible quartic curve which has such singular points. (cf. [N]). □

6. Example

In this section, as an example, we shall see what kind of the singular fibers of rational elliptic surfaces $\mathscr{E}_{C,p}$ show up as the position of $p \in \mathbf{P}^2$ changes, and we determine the OS-numbers of those $\mathscr{E}_{C,p}$. Let C be the irreducible plane quartic curve defined by the equation $F_C(x,y,z) = x^4 + y^4 - xy^2z$. Note that the singularity of C consists of the single point $o = (0:0:1)$, which is $S_{(2,3,3)}$. Also, we remark that C has two double tangent lines, which are $x = z/2$ and $x = -1/2z$.

We first determine the singular fibers by dividing the following cases.

intersection combination	F_v in $\mathscr{E}_{C,p}$	p on C	intersection combination	F_v in $\mathscr{E}_{C,p}$	p on C
N-N-N-N	elliptic curve	—	N^{*1}-$S_{(2,2,2)}$	I_5	I_6
N^{*1}-N-N	I_1	I_2	N^{*1}-$S_{(2,2,3)}$	I_7	I_8
$S_{(1,2,1)}$-N-N	I_3	—	$S_{(1,2,1)}$-$S_{(1,2,1)}$	I_6	—
$S_{(1,2,2)}$-N-N	I_5	—	$S_{(1,2,1)}$-$S_{(1,2,2)}$	I_8	—
$S_{(1,2,3)}$-N-N	I_7	—	$S_{(1,2,1)}$-$S_{(2,2,1)}$	I_5	—
$S_{(2,2,1)}$-N-N	I_2	—	$S_{(1,2,1)}$-$S_{(2,2,2)}$	I_7	—
$S_{(2,2,2)}$-N-N	I_4	—	$S_{(1,2,2)}$-$S_{(2,2,1)}$	I_7	—
$S_{(2,2,3)}$-N-N	I_6	—	$S_{(2,2,1)}$-$S_{(2,2,1)}$	I_4	—
N^{*2}-N	II	III	$S_{(2,2,1)}$-$S_{(2,2,2)}$	I_6	—
$S^*_{(1,2,1)}$-N	IV	—	N^{*3}	III	IV
$S_{(1,3,3)}$-N	IV*	—	$S^*_{(1,2,2)}$	I_1^*	—
$S^*_{(2,2,1)}$-N	III	—	$S^*_{(1,2,3)}$	I_3^*	—
$S_{(2,3,3)}$-N	I_1^*	—	$S^*_{(1,3,3)}$	III*	—
$S_{(3,3,3)}$-N	I_0^*	—	$S^{*2}_{(2,2,1)}$	IV	—
N^{*1}-N^{*1}	I_2	I_3	$S^*_{(2,2,2)}$	I_0^*	—
N^{*1}-$S_{(1,2,1)}$	I_4	I_5	$S^*_{(2,2,3)}$	I_2^*	—
N^{*1}-$S_{(1,2,2)}$	I_6	I_7	$S^{*1}_{(2,3,3)}$	I_2^*	—
N^{*1}-$S_{(1,2,3)}$	I_8	I_9	$S^{*2}_{(2,3,3)}$	IV*	—
N^{*1}-$S_{(2,2,1)}$	I_3	I_4	$S^*_{(3,3,3)}$	I_1^*	—

TABLE 7.

- For the line passing o :
  (i) p is on the x-axis; (ii) p is on the y-axis; (iii) p is not on any line in (i) – (ii).
- For the tangent lines of C :
  (iv) p is one of four points $C \cap \{x = \pm z/2\}$ which are $(1/2 : \pm 1/2 : 1)$ and $(-1/2 : \pm i/2 : 1)$; (v) p is on either line of $x = z/2$ or $x = -z/2$ except four points of (iv); (vi) p is on C except four points of (iv).

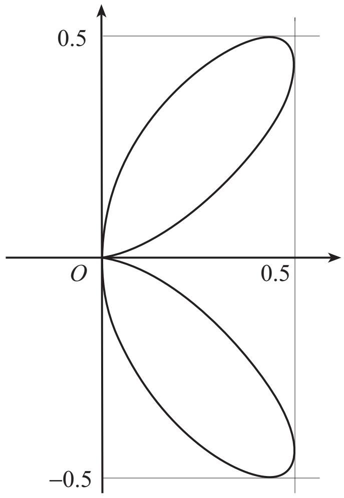

FIGURE 8. Affine part of $z = 1$.

For (i), the intersection combination for the line $\bar{po}$ passing through p and o turns out to be $S^{*2}_{(2,3,3)}$, which corresponds to the singular fiber of type IV* due to Table 7.
For (ii), the intersection combination for the line $\bar{po}$ turns out to be $S^{*1}_{(2,3,3)}$, which corresponds to the singular fiber of type I_2^*.
For (iii), the intersection combination for the line $\bar{po}$ turns out to be $S_{(2,3,3)}$-N, which corresponds to the singular fiber of type I_1^*.
For (iv) and (v), the intersection combination for the double tangent line turns out to be N^{*1}-N^{*1}, which corresponds to the singular fiber of type I_3 and I_4.
For (vi), the intersection combination for the tangent line turns out to be N^{*1}-N-N, which corresponds to the singular fiber of type I_2.

Next we determine the OS-numbers of $\mathscr{E}_{C,p}$ by dividing the following 8 cases :

(1) $p = (x : 0 : z)$ with $(x : z) \neq (0 : 1), (\pm 1/2 : 1)$;
(2) $p = (0 : y : 1)$ with $y \neq 0$;
(3) $p = (1/2 : 0 : 1)$ or $(-1/2 : 0 : 1)$;
(4) $p = (0 : 1 : 0)$;
(5) p is one of four points of (iv) ;
(6) p is a point on either line of $\{x = \pm z/2\}$ except each point of (3),(4) and (5);
(7) p is a point on C except four points of (5);

(8) p is a point different from (1) – (7).

For (1), it follows from (i) that the root lattice of $\mathscr{E}_{C,p}$ is of type E_6, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 27$ by the OS-table.
For (2), it follows from (ii) that the root lattice of $\mathscr{E}_{C,p}$ is of type D_6, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 26$.
For (3), it follows from (i) and (v) that the root lattice of $\mathscr{E}_{C,p}$ is of type $E_6 \oplus A_1$, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 49$.
For (4), it follows from (ii) and (v) that the root lattice of $\mathscr{E}_{C,p}$ is of type $D_6 \oplus A_1^{\oplus 2}$, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 71$. (In fact, $(0:1:0)$ is the intersection point of three lines; y-axis and two double tangent lines.)
For (5), it follows from (iii) and (iv) that the root lattice of $\mathscr{E}_{C,p}$ is of type $D_5 \oplus A_2$, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 50$.
For (6), it follows from (iii) and (v) that the root lattice of $\mathscr{E}_{C,p}$ is of type $D_5 \oplus A_1$, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 30$.
For (7), it follows from (iii) and (vi) that the root lattice of $\mathscr{E}_{C,p}$ is of type $D_5 \oplus A_1$, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 30$.
For (8), it follows from (iii) that the root lattice of $\mathscr{E}_{C,p}$ is of type D_5, therefore $\mathrm{OS\#}(\mathscr{E}_{C,p}) = 16$.

In this way, we can construct various types of elliptic surfaces from just one plane quartic by choosing p suitably.

7. Singular fibers of $\mathscr{E}_{C,p}$ for the reducible case

In this section, we consider the reducible case. If we suppose C is reducible, the degree of each irreducible component of C is less than 4. First we check the singular points which do not appear in the irreducible cases.

Proposition 7.1. *Suppose that a plane quartic C is reducible. Then the singularities of C besides those in Table 2 fall into the following four types.*

sign	s_q	m_q	δ_q
$S_{(2,2,4)}$	2	2	4
$S_{(2,3,4)}$	2	3	4
$S_{(3,3,4)}$	3	3	4
$S_{(4,4,6)}$	4	4	6

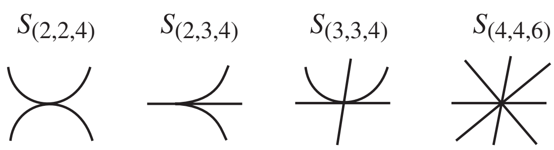

TABLE 9.

Proof. Bertini's theorem implies that $s_q \leq m_q \leq 4$. We proceed a case-by-case analysis on the value s_q. The cases $s_q = 1, 3, 4$ are immediately to prove. In fact, for case $s_q = 1$, q must be a simple cusp, hence this case reduce to Table 2. For $s_q = 3$, there are two possibilities (1) three local branches at q meet transversally or (2) two of local branches coincide. (1) reduces to Table 2 again. For (2), it is verified that $\delta_q = 4$ by the definition of δ (See §2), therefore the singularity is $S_{(3,3,4)}$. For $s_q = 4$, C consists of four lines intersect at one point q, therefore δ_q is 6.

Next we consider the case $s_q = 2$. There are three cases (i) $m_q = 2$, (ii) $m_q = 3$ and (iii) $m_q = 4$. For the case (i), we can assume C consists of two irreducible conics which meet at one point q because otherwise the singularities must be $S_{(2,2,1)}$, which already appeared in Table 2. Since we see that $\delta_q \leq 4$ from some elementary calculations on blowing ups of $C \in \mathbf{P}^2$, we have a singularity $S_{(2,2,4)}$ besides $S_{(2,2,2)}$ and $S_{(2,2,3)}$ Then we obtain one new singularity $S_{(2,2,4)}$. For the case (ii), we can assume C consists of a line and a cubic which has a simple cusp q, and they meet at q. If the line is tangent to the cubic, the singularity is $S_{(2,3,4)}$. If not, it reduces to Table 2. For the case (iii), we immediately see that it is impossible because the degree of each component can not be over 3. □

Here we determine the intersection combination for the above four case by some case-by-case analyses.

(1) The case where C contains no line as an irreducible component.

Note that C consists of two conics, which in fact have neither cusps nor triple points. By using Table 3, it turns out that the intersection combinations and the singular fibers corresponding to them besides those in Table 7 are as in Table 10.

intersection combination	configuration of fiber in $\mathcal{E}_{C,p}$	intersection combination	configuration of fiber in $\mathcal{E}_{C,p}$
$S_{(2,2,4)}$-N-N	I_8	$S_{(2,2,3)}$-$S_{(2,2,1)}$	I_8
$S_{(2,2,2)}$-$S_{(2,2,2)}$	I_8	$S^*_{(2,2,4)}$	I_4^*

TABLE 10.

(2) The case where C contains a line

Let L be a line contained in C. We consider two cases.

(i) p is not on L

In this case, New combinations are only following three cases.

intersection combination	configuration of singular fiber
$S_{(2,3,4)}$-N	III*
$S_{(3,3,4)}$-N	I_2^*
$S_{(4,4,6)}$	—

TABLE 11.

Remark 7.1. If C consists of four lines they meet at one point, the elliptic surface $\mathscr{E}_{C,p}$ has no singular fiber.

(ii) p is on L

In this case L is contained in the branch locus of the double cover φ_{C_0}, and we consider the intersection $L \cap C'$, where $C' := C - L$. Needless to say $L.C' = 3$.)

Proposition 7.2. *The combinations of $L \cap C'$ and the types of singular fiber are as in Table 12.*

intersection combination	configuration of fiber in $\mathscr{E}_{C,p}$	intersection combination	configuration of fiber in $\mathscr{E}_{C,p}$
N-N-N	I_0^*	$S_{(2,2,2)}$-N	I_4^*
N^{*1}-N	I_1^*	N^{*2}	IV*
$S_{(1,2,1)}$-N	I_3^*	$S^*_{(1,2,1)}$	II*
$S_{(2,2,1)}$-N	I_2^*	$S^*_{(2,2,1)}$	III*

TABLE 12.

Thus, if we allow to be reducible, we can construct the singular fiber of I_4^* and II*.

8. The way to distinguish the OS-number of elliptic surfaces with the same root lattices

In §6 and 7, we observed the singular fibers of elliptic surfaces. In fact, we can determine the OS-number from an information of the singular fibers except for the following five pairs (Table 13). For each root lattice of five cases, there are two OS-numbers according to the difference of the Mordell-Weil lattices. In this section, we deal with the way to determine these OS-numbers by the forms of defining equations of E(K).

root lattice T	OS-number	Mordell-Weil-lattice $E(K)$
$A_1^{\oplus 4}$	No.13	$D_4 \oplus \mathbb{Z}/2\mathbb{Z}$
	No.14	$A_1^{*\oplus 4}$
$A_3 \oplus A_1^{\oplus 2}$	No.21	$A_3^* \oplus \mathbb{Z}/2\mathbb{Z}$
	No.22	$A_1^{*\oplus 2} \oplus \langle 1/4 \rangle$
$A_5 \oplus A_1$	No.28	$A_2^* \oplus \mathbb{Z}/2\mathbb{Z}$
	No.29	$A_1^* \oplus \langle 1/6 \rangle$
$A_3^{\oplus 2}$	No.35	$A_1^{*\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$
	No.36	$\langle 1/4 \rangle^{\oplus 2}$
A_7	No.44	$A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$
	No.45	$\langle 1/8 \rangle$

TABLE 13.

Remark 8.1. $\langle 1/m \rangle$ means "a group constructed from a point $P \in E(K)$ such that the height pairing $\langle P, P \rangle$ is $1/m$", which is a non-torsion part (cf.[OS],[Sh]).

In these cases, it is important to observe whether $E(K)$ has torsion parts or not. For each pair, we see that one of Mordell-Weil-lattices has a torsion part $\mathbb{Z}/2\mathbb{Z}$ and the other is torsion free. The following lemma is obvious from the group law algorithm of elliptic curves (see [Si, Chapter III, 2.3]).

Lemma 8.1. *Let $E(k)$ be an elliptic curve given by a Weierstrass equation $E : y^2 z = x^3 + axz^2 + bz^3$ $(a, b \in k)$, and set $o = (0 : 1 : 0)$, which is its extra point at infinity. For a point $P = (x_0, y_0, 1)$, we have $2P = O$ if and only if $y_0 = 0$.*

This lemma immediately yields the following.

Proposition 8.2. *Let $E(k(t))$ be a rational elliptic surface over $\mathbf{P}^1$ given by a Weierstrass equation $y^2z = x^3-27A(t)xz^2-54B(t)z^3$ $(A(t), B(t) \in k(t))$. Then the Mordell-Weil group $E(k(t))$ contains $\mathbb{Z}/2\mathbb{Z}$ if and only if the right hand side is factored to $(x-a(t)z)(x^2-b(t,x,z))$ $(a(t) \in k(t), b(t,x,z) \in k(t,x,z))$.*

Thus, for the five cases in Table 13, we have to find two defining equations of distinct quartics.

9. The main result

The table of the next theorem is the main result.

Theorem 9.1. *For any number $N (1 \leq N \leq 74)$, there exists a pair of a plane quartic curve C and a point p such that $\mathrm{OS}\#(\mathscr{E}_{C,p}) = N$. Furthermore, for each N, an example of (C, p) is given in Table 14 and Weierstrass form $y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$ in Table 15.*

OS-Num	root lattice	example of the equation of C	position of p
No.1	0	$x^4 + y^4 + z^4 = 0$	$(1:1:1)$
No.2	A_1	$x^3y + y^3z + z^3x = 0$	$(0:0:1)$
No.3	A_2	$x^4 + y^4 - z^4 = 0$	$(1:0:1)$
No.4	$A_1 \oplus A_1$	$x^4 + y^4 + x^2z^2 - y^2z^2 = 0$	$(0:1:1)$
No.5	A_3	$x^4 + y^4 + x^3y - x^2z^2 = 0$	$(2:0:1)$
No.6	$A_2 \oplus A_1$	$x^4 + y^4 + y^2z^2 - x^3z = 0$	$(1:0:1)$
No.7	$A_1^{\oplus 3}$	$(x^2 + y^2)^2 - x^2z^2 + y^2z^2 = 0$	$(0:1:1)$
No.8	A_4	$x^3y - x^2z^2 + y^2z^2 = 0$	$(0:1:1)$
No.9	D_4	$x^4 + y^4 - x^2z^2 = 0$	$(0:1:1)$
No.10	$A_3 \oplus A_1$	$x^4 + y^4 + x^3y - x^2z^2 = 0$	$(1:0:1)$
No.11	$A_2 \oplus A_2$	$y^2z^2 - x^3z - x^3y = 0$	$(1:1:1)$
No.12	$A_2 \oplus A_1^{\oplus 2}$	$(x^2 + y^2)^2 - x^2z^2 + y^2z^2 = 0$	$(1:1:1)$
No.13	$A_1^{\oplus 4}$	$x^4 + y^4 - z^4 = 0$	$(0:0:1)$
No.14*	$A_1^{\oplus 4}$	$(x^2 + y^2)^2 - x^2z^2 + y^2z^2 = 0$	$(1:0:1)$
No.15	A_5	$(x^2 + y^2 - 3xz)^2 - 4x^2(2z^2 - xz) = 0$	$(-1:0:1)$
No.16	D_5	$x^4 + y^4 + x^3y - xy^2z = 0$	$(1:1:2)$
No.17	$A_4 \oplus A_1$	$(x^2 - yz)^2 + y(4x^3 - xy^2 - y^3) = 0$	$(0:1:1)$

OS-Num	root lattice	example of the equation of C	position of p
No.18	$D_4 \oplus A_1$	$x^4 + y^4 - x^2yz - xy^2z = 0$	$(1:1:1)$
No.19	$A_3 \oplus A_2$	$x^4 + y^4 - x^2z^2 = 0$	$(1:0:1)$
No.20	$A_2^{\oplus 2} \oplus A_1$	$(x^2 + y^2 - 2xz)^2 - x^2z^2 - y^2z^2 = 0$	$(2:0:1)$
No.21	$A_3 \oplus A_1^{\oplus 2}$	$(2x^2 - z^2 + 2y^2)^2 - 4xy(x-y)^2 = 0$	$(0:0:1)$
No.22*	$A_3 \oplus A_1^{\oplus 2}$	$2x^4 + y^4 - 3x^2yz - 2y^3z + 2y^2z^2 = 0$	$(1:2:1)$
No.23	$A_2 \oplus A_1^{\oplus 3}$	$(3x^2 + y^2)^2 - 6x^2z^2 + 2y^2z^2 = 0$	$(1:1:2)$
No.24	$A_1^{\oplus 5}$	$x^4 + y^4 + x^2z^2 - y^2z^2 = 0$	$(0:1:0)$
No.25	A_6	$x^4 + x^3z - y^2z^2 = 0$	$(0:1:1)$
No.26	D_6	$x^4 + y^4 - xy^2z = 0$	$(0:1:1)$
No.27	E_6	$x^4 + y^4 + x^3y - xy^2z = 0$	$(1:0:2)$
No.28	$A_5 \oplus A_1$	$(z^2 - xz - y^2)^2 - x^3y = 0$	$(0:0:1)$
No.29*	$A_5 \oplus A_1$	$(x^2 + y^2 - 3xz)^2 - 4x^2(2z^2 - xz) = 0$	$(2:0:1)$
No.30	$D_5 \oplus A_1$	$x^4 + y^4 - xy^2z = 0$	$(1:2:2)$
No.31	$A_4 \oplus A_2$	$x^4 + x^3z - y^2z^2 = 0$	$(-1:1:1)$
No.32	$D_4 \oplus A_2$	$x^4 + x^3z - y^2z^2 = 0$	$(1:1:0)$
No.33	$A_4 \oplus A_1^{\oplus 2}$	$y^2(3x^2 - (3z - y)^2)$ $-x^3(9x + 2y - 12z) = 0$	$(0:6:1)$
No.34	$D_4 \oplus A_1^{\oplus 2}$	$x^4 + y^4 - x^2z^2 = 0$	$(0:1:0)$
No.35^r	$A_3^{\oplus 2}$	$(x^2 + y^2 - z^2)(2x^2 + y^2 - 2z^2) = 0$	$(0:2:1)$
No.36*	$A_3^{\oplus 2}$	$x^4 + x^3z + y^2z^2 = 0$	$(2:1:2)$
No.37	$A_3 \oplus A_2 \oplus A_1$	$x^4 + x^3z - 2y^2z^2 = 0$	$(1:1:1)$
No.38	$A_3 \oplus A_1^{\oplus 3}$	$(x^2 + y^2)^2 - x^2z^2 + y^2z^2 = 0$	$(0:1:0)$
No.39	$A_2^{\oplus 3}$	$(x^2 + y^2 - xz)^2 - x^2z^2 - y^2z^2 = 0$	$(1:0:1)$
No.40	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$(x^2 + y^2 - 2xz)^2 - x^2z^2 - y^2z^2 = 0$	$(0:1:1)$
No.41	$A_2 \oplus A_1^{\oplus 4}$	$x^4 + y^4 - y^2z^2 - x^3z = 0$	$(0:1:0)$
No.42	$A_1^{\oplus 6}$	$(z^2 + x^2 + 6xy - 4y^2)^2$ $-12xy(x-y)(x+4y) = 0$	$(0:0:1)$
No.43	E_7	$x^4 - y^3z = 0$	$(1:0:1)$
No.44^r	A_7	$(x^2 - yz)(x^2 + yz) = 0$	$(0:1:1)$
No.45	A_7	$(yz - x^2)^2 - y^3x = 0$	$(0:1:1)$
No.46	D_7	$(yz - x^2)^2 - y^3x = 0$	$(1:0:1)$
No.47	$A_6 \oplus A_1$	$(yz - x^2)^2 - y^3x = 0$	$(1:1:0)$
No.48	$D_6 \oplus A_1$	$x^4 + y^4 + x^3y - xy^2z = 0$	$(0:-1:1)$

OS-Num	root lattice	example of the equation of C	position of p
No.49	$E_6 \oplus A_1$	$x^4 + y^4 - xy^2z = 0$	$(1:0:2)$
No.50	$D_5 \oplus A_2$	$x^4 + y^4 - xy^2z = 0$	$(1:1:2)$
No.51	$A_5 \oplus A_2$	$x^4 + x^3z - y^2z^2 = 0$	$(-1:0:1)$
No.52	$D_5 \oplus A_1^{\oplus 2}$	$x^4 + y^4 - x^2yz - xy^2z = 0$	$(-1:1:0)$
No.53	$A_5 \oplus A_1^{\oplus 2}$	$(x^2 + y^2 - 2xz)^2 - x^2z^2 - y^2z^2 = 0$	$(0:1:0)$
No.54^r	$D_4 \oplus A_3$	$(x^2 - yz)(x^2 + yz) = 0$	$(1:0:1)$
No.55	$A_4 \oplus A_3$	$(x^2 - yz)^2 - x^3y = 0$	$(4:4:-1)$
No.56	$A_4 \oplus A_2 \oplus A_1$	$(x^2 - yz)^2 - x^3y = 0$	$(1:1:0)$
No.57	$D_4 \oplus A_1^{\oplus 3}$	$(x^2 + y^2 - 3xz)^2$ $-4x^2(2z^2 - xz) = 0$	$(0:1:0)$
No.58^r	$A_3^{\oplus 2} \oplus A_1$	$(x^2 + y^2 - z^2)(4x^2 + y^2 - 4z^2) = 0$	$(0:1:1)$
No.59^r	$A_3 \oplus A_2 \oplus A_1^{\oplus 2}$	$(x^2 + 4y^2 - 4z^2)$ $\times(x^2 + (y-z)^2 - 4z^2) = 0$	$(2:1:1)$
No.60	$A_3 \oplus A_1^{\oplus 4}$	$x^4 + y^4 - x^2z^2 = 0$	$(1:0:0)$
No.61	$A_2^{\oplus 3} \oplus A_1$	$(x^2 + y^2 - xz)^2 - x^2z^2 - y^2z^2 = 0$	$(0:1:1)$
No.62^r	E_8	$(x^3 - y^2z)y = 0$	$(1:0:1)$
No.63	A_8	$(yz - x^2)^2 - y^3x = 0$	$(0:1:0)$
No.64^r	D_8	$(x^2 - yz + y^2)(x^2 - yz - y^2) = 0$	$(1:0:1)$
No.65	$E_7 \oplus A_1$	$x^4 - y^3z = 0$	$(1:0:0)$
No.66	$A_5 \oplus A_2 \oplus A_1$	$(x^2 + y^2 - xz)^2 - x^2z^2 - y^2z^2 = 0$	$(0:1:0)$
No.67	$A_4^{\oplus 2}$	$(x^2 - yz)^2 - x^3y = 0$	$(4:16:-1)$
No.68	$A_2^{\oplus 4}$	$(x^2 + 3y^2 - xz)^2$ $-x^2z^2 - 3y^2z^2 = 0$	$(1:1:-4)$
No.69	$E_6 \oplus A_2$	$x^4 - y^3z = 0$	$(0:1:0)$
No.70^r	$A_7 \oplus A_1$	$(x^2 + y^2 - z^2)(y^2 - z^2) = 0$	$(0:0:1)$
No.71	$D_6 \oplus A_1^{\oplus 2}$	$x^4 + y^4 - xy^2z = 0$	$(0:1:0)$
No.72^r	$D_5 \oplus A_3$	$(x^2 + y^2 - z^2)(y^2 - z^2) = 0$	$(1:1:1)$
No.73^r	$D_4^{\oplus 2}$	$(x^2 + y^2 - z^2)(4x^2 + y^2 - 4z^2) = 0$	$(0:1:0)$
No.74^r	$(A_3 \oplus A_1)^{\oplus 2}$	$(x^2 + y^2 - z^2)(y^2 - z^2) = 0$	$(1:0:1)$

TABLE 14.

OS-Num	Equation of elliptic surface constructed from Table 14.
No.1	$y^2 = x^3 + \left(6\,t^4 + 6\right)x^2 + \left(16 + 12\,t^8 - 36\,t^4\right)x$ $+16 + 8\,t^{12} - 24\,t^4 - 96\,t^8$
No.2	$y^2 = x^3 + (3 + 3t^2)x^2 + 4(t^2 - t + 1)(t + 1)^2 x + 2(t^4 + 1)(t + 1)^2$
No.3	$y^2 = x^3 + 3\,x^2 + 4\,x + 2\,t^4 + 2$
No.4	$y^2 = x^3 + \left(5\,t^2 + 1\right)x^2 + 8\,t^4 x + 4\,t^6 + 1$
No.5	$y^2 = x^3 - 4\,t^2 x^2 - 4\,x + 20\,t^2$
No.6	$y^2 = x^3 + \left(3 - t^2\right)x^2 + 3\,x + t^4 + 1$
No.7	$y^2 = x^3 + \left(2 + 14\,t^2\right)x^2 + 32\,\left(2\,t^2 - 1\right)\left(t^2 + 1\right)x$ $+32\,\left(3\,t^2 - 2\right)\left(t^2 + 1\right)^2$
No.8	$y^2 = x^3 + (t - 1)(t + 1)\,x^2 - 2\,t x + 1 + 4\,t$
No.9	$y^2 = x^3 + \left(-1 + 6\,t^2\right)x^2 + \left(12\,t^4 - 4\right)x + 8\,t^6 + 4 + 4\,t^4 - 8\,t^2$
No.10	$y^2 = x^3 + \left(6\,t^4 - 1\right)x^2 + 4\,t^4\left(3\,t^4 - 1\right)x$ $+t^4\left(t^2 + 8\,t^8 + 4\,t - 4\,t^4 + 4\right)$
No.11	$y^2 = x^3 + (-t^2 + 6 + 3t)x^2 + (12t + 12 - 3t^2)x$ $+5t^3 - 15t^2 + 24t + 4$
No.12	$y^2 = x^3 + (8t + 9t^2 + 7)x^2 + 8(1 + t^2)(3t^2 + 8t + 1)x$ $+4(5t^2 + 30t - 3)(1 + t^2)^2$
No.13	$y^2 = x^3 + (4 + 4t^4)x$
No.14*	$y^2 = x^3 + (3t^2 + 5)x^2 + (8t^2 + 8)x + 4(t^2 + 1)^2$
No.15	$y^2 = x^3 + (13 + 8t^2)x^2 + (56 - 16t^4 + 88t^2)x$ $+80 - 224t^2 - 128t^6 + 80t^4$
No.16	$y^2 = x^3 + (-2\,t + 4)\,x^2 + t\,(-5 + t)\,x + 1 - 2\,t + 2\,t^2$
No.17	$y^2 = x^3 - \left(2 + 3\,t + 5\,t^2\right)x^2 + (2\,t + 1)\left(2\,t + 4\,t^3 + 3\,t^2 - 4\right)x$ $+\left(4\,t - t^4 - t^3 + 1\right)(2\,t + 1)^2$
No.18	$y^2 = x^3 + \left(5\,t^2 - 4\,t + 5\right)x^2 + \left(4\,t^2 - 5\,t + 4\right)(t + 1)^2\,x$ $+\left(1 + t^4\right)(t + 1)^2$
No.19	$y^2 = x^3 + 5x^2 + 8x + 4 + 4t^4$
No.20	$y^2 = x^3 + \left(33 + 69\,t^2 + 48\,t\right)x^2 + 144\,\left(t^2 + 1\right)\left(11\,t^2 + 15\,t + 1\right)x$ $+432\,\left(28\,t^2 + 56\,t - 5\right)\left(t^2 + 1\right)^2$
No.21	$y^2 = x^3 - (t^2 + 1)x^2 - (t^4 - t^3 + 4t^2 - t + 1)x$ $+(t^2 + 1)(t^4 - t^3 + 4t^2 + t - 1)$

OS-Num	Equation of elliptic surface constructed from Table 14.
No.22*	$y^2 = x^3 + \left(-6\,t + 6 + 13\,t^2\right) x^2 + (-4 + 9\,t) \left(-3\,t + 6\,t^3 + 8\right) x$ $+ \left(t^4 + 2\right) (-4 + 9\,t)^2$
No.23	$y^2 = x^3 + \left(-t^2 - 12\,t + 9\right) x^2 + 12\,t\,(t + 3)\,(t - 1)\,x$ $-12\,t^2 \left(-3 + 2\,t^2\right)$
No.24	$y^2 = x^3 + t^2x^2 + 4(t - 1)(t + 1)x + 4t^2(t - 1)(t + 1)$
No.25	$y^2 = x^3 + t^2x^2 + (4 - 2\,t)\,x + 1$
No.26	$y^2 = x^3 + 2\,t\,(-1 + 3\,t)\,x^2 + \left(t^2 - 8\,t^3 - 4 + 12\,t^4\right) x$ $-8\,t^5 + 2\,t^4 - 8\,t^2 + 1 + 8\,t^6$
No.27	$y^2 = x^3 - 2\,x^2 + (-4 - t)\,x + 9 + t^2$
No.28	$y^2 = x^3 + \left(-2\,t^2 + 1\right) x^2 - 4\,t \left(t - 1 + t^3\right) x + 4\,t^3 \left(t + 2\,t^3 - 2\right)$
No.29*	$y^2 = x^3 + \left(13 - 4\,t^2\right) x^2 + \left(-8\,t^2 - 16\,t^4 + 56\right) x$ $+80 + 64\,t^6 + 80\,t^4 + 32\,t^2$
No.30	$y^2 = x^3 + (2\,t - 2)\,x^2 - 4\,x + t^2 - 8\,t + 8$
No.31	$y^2 = x^3 + \left(t^2 - 3\right) x^2 + (7 + 6\,t)\,x - 4 - 4$
No.32	$y^2 = x^3 + \left(6 - t^2 + 3\,t\right) x^2 + 3\,(t + 2)^2\,x + 8 + 5\,t^3 + 12\,t + 10\,t^2$
No.33	$y^2 = x^3 + \left(-12 + 13\,t^2\right) x^2 + \left(-96\,t^2 + 56\,t^4 - 144 - 32\,t\right) x$ $+1728 + 80\,t^6 - 128\,t^3 - 576\,t^2 - 192\,t^4 + 384\,t$
No.34	$y^2 = x^3 + 4\,(t - 1)\,(t + 1)\,x$
No.35	$y^2 = x^3 + 6\,\left(21\,t^2 + 8\right) x^2 + 288\,\left(17\,t^4 + 21\,t^2 - 6\right) x$ $+1728\,\left(5\,t^2 + 8\right) \left(7\,t^4 + 5\,t^2 - 6\right)$
No.36*	$y^2 = x^3 + (t + 6)^2\,x^2 + \left(72\,t^2 + 416 + 312\,t\right) x$ $+1552 + 912\,t^2 + 1888\,t$
No.37	$y^2 = x^3 + (9 - 2t^2)x^2 + (35 - 20t)x + (4t - 7)^2$
No.38	$y^2 = x^3 + (2 + t^2)x^2 + 4(t - 1)(t + 1)x + 4(t - 1)(t + 1)(2 + t^2)$
No.39	$y^2 = x^3 + t^2x^2 + 4t^2(t^2 + 1)x + 4t^2(t^2 + 1)^2$
No.40	$y^2 = x^3 + (5 + 5t^2 - 8t)x^2 + 8(t - 1)(t^2 + 1)(t - 2)x$ $+4(t^2 + 1)^2(t - 2)^2$
No.41	$y^2 = x^3 - x^2 - 4\,t^3\,(t - 1)\,x + 4\,t^3\,(t - 1)$
No.42	$y^2 = x^3 + \left(2 + 12\,t - 8\,t^2\right) x^2 - 4\,(2\,t - 1)^2\,(2\,t + 1)^2\,x$ $-8\,\left(1 + 6\,t - 4\,t^2\right) (2\,t - 1)^2\,(2\,t + 1)^2$

OS-Num	Equation of elliptic surface constructed from Table 14.
No.43	$y^2 = x^3 + 6x^2 + \left(12 - 4t^3\right)x + 8 - 8t^3 + t^6$
No.44	$y^2 = x^3 + t^2x^2 + 4x$
No.45*	$y^2 = x^3 + \left(-2 - 3t + t^2\right)x^2 + \left(-t^2 - 4 - 2t^3 + 2t\right)x$ $+9 + 3t^3 + 8t + t^4 + 4t^2$
No.46	$y^2 = x^3 + \left(-4t + 6 + t^2\right)x^2 + \left(2t^4 - 16t + 4t^2 + 12\right)x$ $+4t^4 + 8 + t^6 - 16t + 4t^2$
No.47	$y^2 = x^3 + \left(-4t + 6 + t^2\right)x^2 + 2\,(2t-3)(t-2)x + (2t-3)^2$
No.48	$y^2 = x^3 + 2t\,(3t+1)x^2 + \left(t^2 - 3 + 8t^3 + 12t^4\right)x$ $+(2t+1)\left(2t^2 + t + 2\right)\left(2t^3 - 2t + 1\right)$
No.49	$y^2 = x^3 - 2x^2 - 4x + t^2 + 8$
No.50	$y^2 = x^3 + (-2t+4)x^2 + t\,(t-4)x + t^2$
No.51	$y^2 = x^3 + \left(3 - t^2\right)x^2 + 3x + 1$
No.52	$y^2 = x^3 - 2tx^2 + (t-2)(t+2)x + 2t\,(t+4)$
No.53	$y^2 = x^3 + \left(2 - 4t - t^2\right)x^2 - 4\,(3t-1)(t-1)x$ $+4\,(3t-1)(t-1)\left(-2 + 4t + t^2\right)$
No.54	$y^2 = x^3 + \left(-t^2 + 6\right)x^2 + 12x + 8 + 4t^2$
No.55	$y^2 = x^3 - \left(32t - 56 - t^2\right)x^2 + \left(240t^2 + 960 - 992t\right)x$ $+4864 + 2112t^2 - 6400t$
No.56	$y^2 = x^3 + (t-1)(t-3)x^2 + (2t-1)(2t-3)x + (2t-1)^2$
No.57	$y^2 = x^3 + (-6t+2)x^2 - 4\,(t-1)^2x + 8\,(3t-1)(t-1)^2$
No.58	$y^2 = x^3 + \left(t^2 - 3\right)x^2 - 12t^2\left(5 + 2t^2\right)x + 36t^2\left(1 + t^2\right)\left(4 + t^2\right)$
No.59	$y^2 = x^3 + \left(8t + t^2 + 5\right)x^2 + \left(8 + 8t + 20t^2 + 8t^3\right)x$ $+4\left(1 + 4t^2\right)\left(1 + t^2\right)$
No.60	$y^2 = x^3 - t^2x^2 - 4x + 4t^2$
No.61	$y^2 = x^3 + \left(5t^2 + 2 - 4t\right)x^2 + 4\,(t-1)(2t-1)\left(t^2+1\right)x$ $+4\,(t-1)^2\left(t^2+1\right)^2$
No.62	$y^2 = x^3 + 3tx^2 - t^2\left(t^2 - 3\right)x + t^3$
No.63	$y^2 = x^3 + t^2x^2 + 2tx + 1$
No.64	$y^2 = x^3 + \left(t^2 - 4t + 6\right)x^2 + \left(12 + 4t^4 - 16t + 4t^2\right)x$ $+8t^4 + 8 + 4t^2 - 16t$

OS-Num	Equation of elliptic surface constructed from Table 14.
No.65	$y^2 = x^3 + 4\,t\,x$
No.66	$y^2 = x^3 + \left(-t^2 + 2 - 2\,t\right)x^2 + (8\,t - 4)\,x - 4\,(2\,t - 1)\left(t^2 - 2 + 2\,t\right)$
No.67	$y^2 = x^3 + \left(-32\,t - 64 + t^2\right)x^2 + 2560\,t\,x + 65536 - 65536\,t$
No.68	$y^2 = x^3 - 3\,(t+6)(t-2)\,x^2 - 96\,(t+4)(t-2)\,x$ $-256\,(2\,t-1)(t+4)^2$
No.69	$y^2 = x^3 + t^2$
No.70	$y^2 = x^3 + \left(-1 - 2\,t^2\right)x^2 - 4\,t^2\left(t^2+1\right)x + 4\,t^2\left(2\,t^2+1\right)\left(t^2+1\right)$
No.71	$y^2 = x^3 - t\,x^2 - 4\,x + 4\,t$
No.72	$y^2 = x^3 + t\,(4 + 5\,t)\,x^2 + 4\,t^2\left(2\,t^2 + t + 1\right)x + 4\,t^4\left(t^2+1\right)$
No.73	$y^2 = x^3 - 5\,(t-1)(t+1)\,x^2 - 16\,(t-1)^2(t+1)^2\,x$ $+80\,(t-1)^3(t+1)^3$
No.74	$y^2 = x^3 + \left(-t^2 - 1\right)x^2 - 4\,t^2 x + 4\,t^2\left(t^2+1\right)$

TABLE 15.

Remark 9.1. the symbol ” $*$ ” indicates that the elliptic surfaces have their same root lattices. The symbol ”r” means that the plane quartics are reducible.

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DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
3-4-1 OHKUBO, SHINJUKU, TOKYO 169-8555, JAPAN
E-mail address: d-i-waseda@akane.waseda.jp