# 「応用問題」略解

[1] ～ [3] は授業でやったはず（だと思う）.

[4] 車の移動をを表す行列を求めると $\begin{pmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.6 \\ 0.1 & 0.5 & 0.2 \end{pmatrix}$ となる．この行列の固有値１の固有ベクトルを求めると $t\begin{pmatrix} 34 \\ 14 \\ 13 \end{pmatrix}$ となる．したがって，各営業所の車の台数の割合は 34 : 14 : 13 となる．そこで，各営業所に 34 : 14 : 13 の割合で配車しておけば，いつも同じ割合の車が各営業所にあることになる.

[5] a) 固有値 0.5；固有ベクトル $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$，固有値 1.5；固有ベクトル $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b) $P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$ とおくと，$P^{-1}AP = \begin{pmatrix} 0.5 & 0 \\ 0 & 1.5 \end{pmatrix}$

c) $A^n = P\begin{pmatrix} 0.5^n & 0 \\ 0 & 1.5^n \end{pmatrix}P^{-1} = \frac{1}{5}\begin{pmatrix} 4\cdot 0.5^n + 1.5^n & -2\cdot 0.5^n + 2\cdot 1.5^n \\ -2\cdot 0.5^n + 2\cdot 1.5^n & 0.5^n + 4\cdot 1.5^n \end{pmatrix}$

d) $n$ が大きくなると $0.5^n$ は急速に 0 に近づく．このとき，$A^n$ は c) で得た $A^n$ の式で $0.5^n$ を 0 に置き換えたもので近似できる．すなわち，

$$A^n \fallingdotseq \frac{1}{5}\begin{pmatrix} 1.5^n & 2\cdot 1.5^n \\ 2\cdot 1.5^n & 4\cdot 1.5^n \end{pmatrix} = \frac{1.5^n}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

となる．これは郊外人口と都市人口との比率が 1 : 2 であり，また，そのときの都市圏全体の人口は毎年おおよそ 1.5 倍増加することを示す.

[6] Let $a_n, b_n, c_n$ be the probability that the $n$th day is nice day, rainy day, or snowy day respectively. We have the relation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}.$$

Write $A = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$, and $\vec{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$.

a) $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ としたとき，$\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}$ を求めればよい．

$$\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = A^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.375 \\ 0.375 \end{pmatrix}$$

b) First, diagonalize $A$. Put $P = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$. Then we have

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & -0.25 \end{pmatrix}.$$

Thus, we have

$$A^n = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.25^n & 0 \\ 0 & 0 & (-0.25)^n \end{pmatrix}P^{-1}$$
$$= \frac{1}{10}\begin{pmatrix} 2+8\cdot(-0.25)^n & 2-2\cdot(-0.25)^n & 2-2\cdot(-0.25)^n \\ 4-4\cdot(-0.25)^n & 4+5\cdot 0.25^n+(-0.25)^n & 4-5\cdot 0.25^n+(-0.25)^n \\ 4-4\cdot(-0.25)^n & 4-5\cdot 0.25^n+(-0.25)^n & 4+5\cdot 0.25^n+(-0.25)^n \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = A^{n-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2+8\cdot(-0.25)^{n-1} \\ 4-4\cdot(-0.25)^{n-1} \\ 4-4\cdot(-0.25)^{n-1} \end{pmatrix}.$$

c) Consider the limit $n \to \infty$.

$$\lim_{n\to\infty}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \lim_{n\to\infty} A^{n-1}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$
$$= \frac{a_1+b_1+c_1}{10}\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}.$$

Here, we used the fact $a_1+b_1+c_1 = 1$. Thus, in the land of Oz, it is fair 20% of the time, rainy 40% of the time, and snowy 40% of the time, in general.

[7] Let $a_n, b_n, c_n$ be the populations of region 1, 2, 3, respectively, after $n$ years from now. Of the $b_n$ people living in region 2, $0.15\,b_n$ move to region 1 in the following year, and of the $c_n$ people living in region 3, $0.10\,c_n$ move to region 1. Together with the $0.90\,a_n$ people remaining in region 1, the population of region 1 in the following year $a_{n+1}$ equals $0.90\,a_n + 0.15\,b_n + 0.10\,c_n$. Similarly, we can find the population of region 2 and 3 in the following year, and we obtain the relation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0.90 & 0.15 & 0.10 \\ 0.05 & 0.75 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}.$$

Write $A = \begin{pmatrix} 0.90 & 0.15 & 0.10 \\ 0.05 & 0.75 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}$, and $\vec{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$. What we want to find is

$$\lim_{n\to\infty}\vec{v}_n = \lim_{n\to\infty} A^n\vec{v}_0.$$

If the limit $\vec{v}_\infty = \lim_{n\to\infty} \vec{v}_n$ exists, it should satisfy $A\vec{v}_\infty = \vec{v}_\infty$. In other words, the limit $\vec{v}_\infty$ is an eigenvector of $A$ belonging to the eigenvalue 1.

If we calculate the characteristic polynomial of $A$, then we have $\det(A-\lambda I) = -(\lambda-1)(\lambda-4/5)(\lambda-7/10)$. Thus, $A$ does have the eigenvalue 1. Solving $(A-\lambda I)\vec{x} = \vec{0}$, we find that the eigenvectors belonging to the eigenvalue 1 are $t\begin{pmatrix}13\\4\\7\end{pmatrix}$ ($t$ is an arbitrary real number). Thus, once we show the existence of the limit, we can conclude that the populations of region 1, 2, 3 will be $13/24 \approx 54.2\%$, $4/24 \approx 16.7\%$, $7/24 \approx 29.2\%$, respectively.

To show that the limit exists, we look for the eigenvectors belonging to $4/5$ and $7/10$. Solving the linear equations, we have

$$\text{eigenvalue } \frac{4}{5},\ \text{eigenvector } t\begin{pmatrix}-1\\0\\1\end{pmatrix}; \quad \text{eigenvalue } \frac{7}{10},\ \text{eigenvector } t\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

From this we have

$$\begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0.90 & 0.15 & 0.10\\0.05 & 0.75 & 0.05\\0.05 & 0.10 & 0.85\end{pmatrix}\begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 4/5 & 0\\0 & 0 & 7/10\end{pmatrix}.$$

Taking the $n$-th power of both sides, we have

$$\begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0.90 & 0.15 & 0.10\\0.05 & 0.75 & 0.05\\0.05 & 0.10 & 0.85\end{pmatrix}^{n}\begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & (4/5)^n & 0\\0 & 0 & (7/10)^n\end{pmatrix}.$$

From this we can prove that

$$\lim_{n\to\infty} A^n = \begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}13 & -1 & 1\\4 & 0 & -2\\7 & 1 & 1\end{pmatrix}^{-1} = \frac{1}{24}\begin{pmatrix}13 & 13 & 13\\4 & 4 & 4\\7 & 7 & 7\end{pmatrix}.$$

Thus, the limit $\vec{v}_\infty$ exists no matter what the initial vector $\vec{v}_0$ is.